Ruin probability in the presence of risky investments.*

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Abstract

We consider an insurance company in the case when the premium rate is a bounded non-negative random function c_t and the capital of the insurance company is invested in a risky asset whose price follows a geometric Brownian motion with mean return a and volatility $\sigma > 0$. If $\beta := 2a/\sigma^2 - 1 > 0$ we find exact the asymptotic upper and lower bounds for the ruin probability $\Psi(u)$ as the initial endowment u tends to infinity, i.e. we show that $C_* u^{-\beta} \le \Psi(u) \le C^* u^{-\beta}$ for sufficiently large u. Moreover if $c_t = c^* e^{\gamma t}$ with $\gamma \le 0$ we find the exact asymptotics of the ruin probability, namely $\Psi(u) \sim u^{-\beta}$. If $\beta \le 0$, we show that $\Psi(u) = 1$ for any $u \ge 0$.

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1 Introduction

It is well known that the analysis of activity of an insurance company in conditions of uncertainty is of great importance. Starting from the classical papers of Cramér and Lundberg which first considered the ruin problem in stochastic environment, this subject has attracted much attention. Recall that, in the classical Cramér–Lundberg model satisfying the Cramér condition and, the positive safety loading

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assumption, the ruin probability as a function of the initial endowment decreases exponentially (see, for example, Mikosch [12]). The problem was subsequently extended to the case when the insurance risk process is a general Lévy process (see, for example, Klüppelberg et al. [10] for details).

More recently ruin problems have been studied in application to an insurance company which invests its capital in a risky asset see, e.g., Paulsen [14], Kalshnikov and Norberg [9], Frolova et al. [5] and many others.

It is clear that, risky investment can be dangerous: disasters may arrive in the period when the market value of assets is low and the company will not be able to cover losses by selling these assets because of price fluctuations. Regulators are rather attentive to this issue and impose stringent constraints on company portfolios. Typically, junk bonds are prohibited and a prescribed (large) part of the portfolio should contain non-risky assets (e.g., Treasury bonds) while in the remaining part only risky assets with good ratings are allowed. The common notion that investments in an asset with stochastic interest rate may be too risky for an insurance company can be justified mathematically.

We deal with the ruin problem for an insurance company investing its capital in a risky asset specified by a geometric Brownian motion

$$\mathrm{d}V_t = V_t(a\mathrm{d}t + \sigma \mathrm{d}w_t)\,, \tag{1.1}$$

where $(w_t,\, t \geq 0)$ is a standard Brownion motion and $a > 0$, $\sigma > 0$.

It turns out that in this case of *small volatility*, i.e. $0 < \sigma^2 < 2a$, the ruin probability is not exponential but a power function of the initial capital with the exponent $\beta := 2a/\sigma^2 - 1$. It will be noted that this result holds without the requirement of positive safety loading. Also, for large volatility, i.e. $\sigma^2 > 2a$, the ruin probability equals 1 for any initial endowment. These results have been obtained under various conditions in [14, 9, 5].

Additionally, a large deviations limiting theorems for describing the ruin probability was obtained by Djehiche [3] and Nyrhinen [13]. Gaier et al. [6] studied the optimal investment problem for an insurance company.

In all these papers the premium rate was assumed to be constant. In practice this means that the company should obtain a premium with the same rate continuously. We think that this condition is too restrictive and it significantly bounds the applicability of the above mentioned results in practical insurance settings.

The goal of this paper is to consider the ruin problem for an insurance company for which the premium rate is specified by a bounded non-negative random function c_t. For the given problem, under the condition of *small volatility*, we derive exact upper and lower bounds for the ruin probability and in the case of exponential premium rate, i.e. $c_t = e^{\gamma t}$ with $\gamma \leq 0$, we find the exact asymptotics for the ruin

probability. Particularly, we show that for the zero premium rate, i.e. $\gamma = -\infty$, the asymptotic result is the same as in the case $-\infty < \gamma < 0$.

Moreover, in this paper we show that in the boundary case, i.e. $\sigma^2 = 2a$, the company goes bankrupt with probability 1 for any bounded function c_t.

Indeed, an upper bound for the ruin probability for the random function c_t in the small volatility case is obtained also by Ma and Sun [11].

The paper is organised as follows. In the next section we give the main results. In Section 3 we give the necessary results about the tails of solutions of some linear random equation which we apply to study the ruin problem. In Section 4 we obtain the upper bound for the ruin probability and in Section 5 we find the corresponding lower bound. In Section 6 we consider the exponential premium income rate case. In Section 7 we study some ergodic properties for an autoregressive process with random coefficient. in Section 8 we consider the large volatility case.

2 Basic results

Let us consider a process $X = X^u$ of the form

$$X_t = u + a\int_0^t X_s \mathrm{d}s + \sigma \int_0^t X_s \mathrm{d}w_s + \int_0^t c_s \mathrm{d}s - \sum_{i=1}^{N_t} \xi_i \,, \qquad (2.1)$$

where $a \geq 0$ and $\sigma \geq 0$ are arbitrary constants, w is a Brownion motion, N is a Poisson process with intensity $\alpha > 0$ and $(\xi_i \,, i \in \mathbb{N})$ are i.i.d. positive random variables with common a distribution F. Moreover, we assume that w, N, (ξ_i) are independent and the filtration is defined as $\mathcal{F}_t = \sigma \left\{ w_s \,, N_s \,, \sum_{i=1}^{N_s} \xi_i \,, 0 \leq s \leq t \right\}$. Furthermore, $c_t = c(t, X)$ is a bounded non-negative $(\mathcal{F}_t)$ - adapted function (i.e., $0 \leq c_t \leq c^*$) such that Eq. (2.1) has an unique strong solution (see chapter 14 in [8]).

Let $\varsigma_u := \inf\{t : X_t^u < 0\}$ (the time of ruin), $\Psi(u) := P(\varsigma_u < \infty)$ (the ruin probability). The parameter values $a = 0$, $\sigma = 0$, $c_t = c$, correspond to the Cramér–Lundberg model for which the risk process is usually written as $X_t = u + c\,t - \sum_{i=1}^{N_t} \xi_i$. In the considered version (of non-life insurance) the capital evolves due to a continuously incoming cash flow with rate $c > 0$ and outgoing random payoffs ξ_i at times forming an independent Poisson process N with intensity α. For the model with positive safety loading and F having a ”non-heavy” tail, the Lundberg inequality provides encouraging information: the ruin probability decreases exponentially as the initial endowment u tends to infinity. Moreover, for exponentially distributed claims the ruin probability admits an explicit expression, see [1] or [12].

We study here the case $\sigma > 0$ with a general random adapted bounded function c_t. In this case Eq. (2.1) describes the evolution of the capital of an insurance company, which is continuously reinvested into an asset with the price following a geometric Brownian motion (1.1).

Let $\beta := 2a/\sigma^2 - 1$. To write the upper bound for the ruin probability we define the function :

$$J(\beta) = \frac{2\alpha}{\sigma^2 \beta^2} \left(\mathbf{1}_{\{0<\beta\le 1\}} + j_1(\beta)\, \mathbf{1}_{\{1<\beta\le 2\}} + j_2(\beta)\, \mathbf{1}_{\{\beta>2\}} \right) , \tag{2.2}$$

where $j_1(\beta) = \beta\,(1+\varrho^{-1})$, $j_2(\beta) = \beta\, 2^{\beta-2}(1+((1+\varrho)^{\frac{1}{\beta-1}}-1)^{1-\beta})$ and $\varrho = \varrho(\beta) = (\beta-1)\sigma^2/2\alpha$.

Theorem 2.1. *If $\beta > 0$ and $\mathbf{E}\,\xi_1^\beta < \infty$, then $\limsup_{u\to+\infty} u^\beta \Psi(u) \le C^*(\beta)$, where $C^*(\beta) = J(\beta)\mathbf{E}\,\xi_1^\beta$.*

The proof of this theorem is given in Section 4.

Theorem 2.2. *If $\beta > 0$ and $\mathbf{E}\xi_1^{\beta+\delta} < \infty$ for some $\delta > 0$, then there exists a constant $0 < C_* < \infty$ such that $\liminf_{u\to\infty} u^\beta \Psi(u) \ge C_*$.*

This result is proved in Section 5. The following theorem gives the exact asymptotics for the exponential function c_t.

Theorem 2.3. *Assume that $c_t = c^* \exp\{\gamma t\}$ with $-\infty \le \gamma \le 0$. If $\beta > 0$ and $\mathbf{E}\,\xi_1^{\beta+\delta} < \infty$ for some $\delta > 0$, then there exists a constant $0 < C_\infty < \infty$ such that $\lim_{u\to\infty} u^\beta\, \Psi(u) = C_\infty$. Moreover, the constant C_∞ is the same for any $-\infty \le \gamma < 0$.*

This result is proved in Section 6. Now we consider the large volatility case, i.e. $\beta \le 0$.

Theorem 2.4. *Assume that the distribution of ξ_1 has not a finite support, i.e. $\mathbf{P}(\xi_1 > z) > 0$ for any $z \in \mathbb{R}$. If $\beta \le 0$ and $\mathbf{E}\xi_1^\delta < \infty$ for some $\delta > 0$, then $\Psi(u) = 1$ for any $u \ge 0$.*

Remark 2.5. *This theorem has been proved by Paulsen in [14] for a constant premium rate, i.e. for $c_t = c^* = const$.*

The key idea in the proofs of Theorem 2.1 and Theorem 2.2 is based on the fact that the function $\Psi(u)$ may be estimated by the tails of solutions of some linear random equations. In the next section we study the asymptotic behaviour of those tails.

3 Tails of solutions of random equations

This Section contains some results from the general renewal theory developed by Goldie [7] for some random equations. We consider the following two random equations

$$R \overset{(d)}{=} Q + M\,R\,, \quad R \quad \text{is independent of} \quad (M,Q) \tag{3.1}$$

($\overset{(d)}{=}$ denoting equality of probability laws) and

$$R^* \overset{(d)}{=} Q + M\,(R^*)_+\,, \quad R^* \quad \text{independent of} \quad (M,Q)\,, \tag{3.2}$$

where $(a)_+ = \max(a,0)$.

We start with some preliminary conditions for the random variable M which are studied by Goldie (see Lemma 2.2 in [7]).

Lemma 3.1. *Let $M \geq 0$ be a random variable such that, for some $\beta > 0$*

$$\mathbf{E}\,M^\beta\, = 1\,, \quad \mathbf{E}\,M^\beta\,(\log\,M)_+ < \infty \tag{3.3}$$

and the conditional law of $\log\,M$*, given $M \neq 0$, be non-arithmetic. Then $-\infty \leq \log \mathbf{E} M < 0$ and $0 < \mu := \mathbf{E} M^\beta \log M < \infty$.*

The following result from [7] specifies the tail behaviour of R.

Lemma 3.2. *(Theorem 4.1 in [7]) Let M be a random variable satisfying the conditions of Lemma 3.1 for some $\beta > 0$ and Q be a positive random variable for which $\mathbf{E}\,Q^\beta < \infty$. Then there is a unique law for R satisfying* (3.1) *such that*

$$\lim_{u\to+\infty} u^\beta\,\mathbf{P}(R > u) = c_\infty\,, \tag{3.4}$$

where $c_\infty = \mathbf{E}\,\left((Q+MR)_+^\beta - \,(MR)_+^\beta\right)/\beta\,\mu$ and $\mu = \mathbf{E}\,M^\beta \log\,M$.

Now we study the tail of R^*.

Lemma 3.3. *Let $M \geq 0$ be a random variable satisfying the conditions of Lemma 3.1 for some $\beta > 0$. Assume also that the distribution of M is absolutely continuous with respect to Lebesgue measure and there exists $\delta > 0$ such that*

$$\mathbf{E}\,M^{\beta+\delta} < \infty \tag{3.5}$$

and for any $x \in \mathbb{R}$

$$\mathbf{E}\,M^{\beta+\delta+ix} \neq 1\,, \tag{3.6}$$

where $i = \sqrt{-1}$. *Then under the condition*

$$\mathbf{E}\,|Q|^{\beta+\delta} < \infty \tag{3.7}$$

for some $\delta > 0$ *there is a unique law for* R^* *satisfying* (3.2) *such that there exists* $\lim_{u\to\infty} u^{\beta}\mathbf{P}(R^* > u) = c^*_\infty$ *and* $0 < c^*_\infty < \infty$.

This lemma follows directly from Theorem 6.3 in [7] and Theorem 2 in [13].

4 Upper bound for the ruin probability

Let τ_n be the instant of n-th jump of N and let $\theta_n := \tau_n - \tau_{n-1}$ with $\tau_0 := 0$. We define the discrete-time process $S = S^u$ with $S_n := X_{\tau_n}$. Since ruin may occur only when X jumps downwards, $\Psi(u) = P(T_u < \infty)$, where

$$T_u := \inf\{n \ge 1 :\ S_n < 0\}. \tag{4.1}$$

Therefore to obtain asymptotic properties of T_u as $u \to \infty$ we need to study the process (S_n). First of all, we need to find a recurrence equation for this sequence. We start with resolving of Eq. (2.1). For this we introduce the process $(\phi^{s,x}_t)_{t\ge s}$ which satisfies the following stochastic differential equation

$$\mathrm{d}\phi^{s,x}_t \,=\, a\,\phi^{s,x}_t\,\mathrm{d}t \,+\, \sigma\,\phi^{s,x}_t\,\mathrm{d}w_t \,+\, c_t\,\mathrm{d}t\,,\quad \phi^{s,x}_s = x\,.$$

The Ito formula implies that $\phi^{s,x}_t \,=\, e^{h_t-h_s}\,x \,+\, \int_s^t\, e^{h_t-h_u}\,c_u\,\mathrm{d}u$, where $h_t = \kappa\,t + \sigma w_t$, $\kappa = a - \sigma^2/2$ and $t \ge s$. Moreover we can represent Eq. (2.1) for $\tau_{n-1} \le t < \tau_n$ in the following way

$$\begin{aligned} X_t \,=&\, S_{n-1} \,+\, a\int_{\tau_{n-1}}^t X_s\mathrm{d}s + \sigma\int_{\tau_{n-1}}^t X_s\mathrm{d}w_s + \int_{\tau_{n-1}}^t c_s\mathrm{d}s \,=\, \phi^{\tau_{n-1},S_{n-1}}_t \\ =&\, e^{h_t - h_{\tau_{n-1}}}\,S_{n-1} \,+\, \int_{\tau_{n-1}}^t\, e^{h_t-h_u}\,c_u\,\mathrm{d}u\,. \end{aligned}$$

Therefore $S_n = X_{\tau_n} = \phi^{\tau_{n-1},S_{n-1}}_{\tau_n} - \xi_n$. From this we obtain the following random recurrence equation for (S_n)

$$S_n \,=\, \lambda_n\,S_{n-1} \,+\, \zeta_n\,,\quad S_0 = u \tag{4.2}$$

with $\lambda_n = \exp\{\sigma\,w^n_{\theta_n} + \kappa\theta_n\}$ and $\zeta_n = \eta_n - \xi_n$. Here $w^n_t = w_{t+\tau_{n-1}} - w_{\tau_{n-1}}$ and $\eta_n = \int_0^{\theta_n} c^n_u e^{h_{\tau_n} - h_{u+\tau_{n-1}}}\mathrm{d}u$ with $c^n_u := c_{u+\tau_{n-1}}$. By resolving (4.2) we find the following representation for (S_n)

$$S_n \,=\, \mathcal{E}_n\,u + \mathcal{E}_n\sum_{k=1}^n \mathcal{E}^{-1}_k\,\zeta_k\,,\quad \mathcal{E}_n \,=\, \prod_{k=1}^n \lambda_k\,. \tag{4.3}$$

Moreover, taking into account here that $\zeta_k \geq -\xi_k$ we obtain that $S_n \geq \mathcal{E}_n\,(u - Y_n)$, where

$$Y_n = Q_1 + \sum_{k=2}^{n} Q_k \prod_{j=1}^{k-1} M_j\,, \quad M_j = \lambda_j^{-1}\,, \quad Q_k = \xi_k/\lambda_k\,. \tag{4.4}$$

Notice that (M_n) are i.i.d. random variables such that for $q \in]0, \beta]$

$$\mathbf{E}\, M_1^q \;=\; \mathbf{E}\, \lambda_1^{-q} = \frac{2\alpha}{2\alpha + (\beta - q)\, q\, \sigma^2} \leq 1\,. \tag{4.5}$$

Therefore, there exists $0 < \delta < \min(1, \beta)$ for which $\rho = \mathbf{E}\, M_1^\delta < 1$ and

$$\mathbf{E}\, \left(\sum_{k\geq 2} Q_k \prod_{j=1}^{k-1} M_j \right)^{\delta} \;\leq\; \sum_{k\geq 2} \mathbf{E}\, \left(Q_k \prod_{j=1}^{k-1} M_j \right)^{\delta} = \mathbf{E}\, Q_1^\delta \sum_{k\geq 2} \rho^{k-1} \;<\; \infty\,,$$

i.e. the series $\sum_{k\geq 2} Q_k \prod_{j=1}^{k-1} M_j$ is finite a.s. It means that the sequence (Y_n) have a finite limit

$$\lim_{n\to\infty} Y_n \;=\; Q_1 + \sum_{k=2}^{+\infty} Q_k \prod_{j=1}^{k-1} M_j = Y_\infty = R \;<\; \infty \quad \text{a.s.} \tag{4.6}$$

Taking into account that the sequence (Y_n) in (4.4) is increasing we can estimate S_n as

$$S_n \;\geq\; \mathcal{E}_n\,(u - R) \tag{4.7}$$

and by (4.1) we get that $\mathbf{P}(T_u < \infty) \leq \mathbf{P}(R > u)$. Therefore, to obtain the upper bound for the ruin probability we investigate the tail behaviour of R as $u \to \infty$. To this end, first notice that we may represent R in the following form

$$R = Q_1 + M_1\, R_1\,, \tag{4.8}$$

where the random variable $R_1 \,=\, Q_2 + \sum_{k=3}^{+\infty} \prod_{j=2}^{k-1} M_j\, Q_k$ has the same distribution as R and is independent of (Q_1, M_1). Thus the random variable R satisfies Eq. (3.1). We show that

$$\lim_{u\to\infty} u^\beta\, \mathbf{P}(R > u) = C_1\,, \tag{4.9}$$

where $C_1 = 2\alpha \mathbf{E}\,((\xi_1 + R)^\beta - R^\beta)/\beta^2\, \sigma^2$.

To show (4.9) we need to check the conditions of Lemma 3.2 for the random variables (M_j) and (Q_j) defined in (4.4). The first property in (3.3) follows directly from (4.5) for $q = \beta$. Now we show the second. By definition of M_1 we have

$$\begin{aligned}\mathbf{E}\, M_1^\beta\, (\log\, M_1)_+ &= \mathbf{E}\, e^{-\beta\,\sigma\, w_{\theta_1} - \beta\,\kappa\,\theta_1}\, (-\sigma\, w_{\theta_1} - \kappa\,\theta_1)\, \mathbf{1}_{\{-\sigma\, w_{\theta_1} - \kappa\,\theta_1 \geq 0\}} \\ &\leq\; \sigma\, \mathbf{E}\, |w_{\theta_1}|\, e^{-\beta\,\sigma\, w_{\theta_1} - \beta\,\kappa\,\theta_1} \;+\; \kappa\, \mathbf{E}\, \theta_1\, e^{-\beta\,\sigma\, w_{\theta_1} - \beta\,\kappa\,\theta_1}\,.\end{aligned}$$

Taking into account that (w_t) is independent of (θ_j), the last term in this inequality equals

$$\sigma\,\frac{1}{\sqrt{2\pi}}\mathbf{E}\,\sqrt{\theta_1}\,\int_{-\infty}^{+\infty}\,|z|\,e^{-(z+\beta\,\sigma\,\sqrt{\theta_1})^2/2}\,\mathrm{d}z+\,\kappa\,\mathbf{E}\,\theta_1\,,$$

i.e. $\mathbf{E}\,M_1^\beta\,(\log\,M_1)_+\le\,(\beta\sigma^2+\kappa)\,\mathbf{E}\,\theta_1+\sigma\,\sqrt{2/\pi}\,\mathbf{E}\,\sqrt{\theta_1}<\infty$. In similar way we calculate $\mu=\mathbf{E}M_1^\beta\log\,M_1=\beta\sigma^2/2\alpha$. Moreover, $\mathbf{E}\,Q_1^\beta=\mathbf{E}\,\xi_1^\beta<\infty$. Therefore, by making use of Lemma 3.2 we get the limiting relationship (4.9) which implies that $\limsup_{u\to\infty}\,u^\beta\Psi(u)\le C_1$. Thus, to finish the proof we need to show the inequality $C_1\le C^*(\beta)$. Indeed, if $0<\beta\le 1$, then $\mathbf{E}((\xi_1+R)^\beta-R^\beta)\le\mathbf{E}\xi_1^\beta$ and, therefore, in this case $C_1\le C^*(\beta)$. If $\beta>1$, then, taking into account the inequality $a^\beta-b^\beta\le\beta\,(a-b)\,a^{\beta-1}$ ($0<b<a$), we obtain that $C_1\le 2\alpha\mathbf{E}\xi_1(\xi_1+R)^{\beta-1}/\beta\sigma^2$. This implies that for $1<\beta\le 2$,

$$C_1\le\frac{2\alpha}{\beta\,\sigma^2}\,(\mathbf{E}\,\xi_1^\beta+\mathbf{E}\,\xi_1\,\mathbf{E}\,R^{\beta-1})\le\frac{2\alpha}{\beta\,\sigma^2}\,(\mathbf{E}\,\xi_1^\beta+(\mathbf{E}\,\xi_1^\beta)^{\frac{1}{\beta}}\,\mathbf{E}\,R^{\beta-1})\,. \qquad (4.10)$$

Since by (4.5) we have $\mathbf{E}\,M_1^{\beta-1}\,<1$, therefore by making use of (4.8) and taking into account that $(\mathbf{E}\,M_1^{\beta-1})^{-1}-1=\varrho$ (ϱ is defined in (2.2)) we can estimate $\mathbf{E}\,R^{\beta-1}$ as

$$\mathbf{E}\,R^{\beta-1}\le\frac{\mathbf{E}\,Q_1^{\beta-1}}{1-\mathbf{E}\,M_1^{\beta-1}}=\frac{\mathbf{E}\,\xi_1^{\beta-1}\,\mathbf{E}\,M_1^{\beta-1}}{1-\mathbf{E}\,M_1^{\beta-1}}\le\frac{1}{\varrho}\,(\mathbf{E}\,\xi^\beta)^{\frac{\beta-1}{\beta}}\,.$$

Thus, from this and (4.10), we obtain that $C_1\le C^*(\beta)$ for $1<\beta\le 2$. Let us consider now the case $\beta>2$. In this case we estimate C_1 as

$$C_1\le\frac{2^{\beta-1}\alpha}{\beta\sigma^2}(\mathbf{E}\xi_1^\beta+\mathbf{E}\xi_1R^{\beta-1})\le\frac{2^{\beta-1}\alpha}{\beta\,\sigma^2}(\mathbf{E}\xi_1^\beta+(\mathbf{E}\xi_1^\beta)^{\frac{1}{\beta}}\mathbf{E}R^{\beta-1})\,. \qquad (4.11)$$

We set $\|R\|_q=(\mathbf{E}\,R^q)^{\frac{1}{q}}$ with $q=\beta-1$. Taking into account that the random variables R_1 and M_1 are independent in (4.8), we obtain that

$$\|R\|_q\,=\,\|M_1\,R_1\,+\,Q_1\|_q\le\|M_1\|_q\,\|R_1\|_q\,+\,\|Q_1\,\|_q\,,$$

i.e. $\|R\|_q\le\|Q_1\|_q(1-\|M_1\|_q)^{-1}=\|\xi_1\|_q((\|M_1\|_q)^{-1}-1)^{-1}$. From this, we find

$$\mathbf{E}\,R^{\beta-1}\le\,\Big((1+\varrho)^{\frac{1}{\beta-1}}-1\Big)^{1-\beta}\,(\mathbf{E}\,\xi_1^\beta)^{\frac{\beta-1}{\beta}}\,.$$

Applying this inequality to (4.11), one obtains $C_1\le C^*(\beta)$ for $\beta>2$. This implies Theorem 2.1. $\square$

5 Lower bound for the ruin probability

In this section we prove Theorem 2.2. First, notice that the identity (4.3) implies

$$S_n \le S_n^* := \mathcal{E}_n\, u + \mathcal{E}_n \sum_{k=1}^{n} \mathcal{E}_k^{-1}\, \zeta_k^*\,, \tag{5.1}$$

where $\zeta_k^* = \eta_k^* - \xi_k$ with $\eta_k^* := c^* \int_0^{\theta_k} e^{h_{\tau_k} - h_{u+\tau_{k-1}}} \mathrm{d}u$. Therefore, denoting $T_u^* = \inf\{n \ge 1 \,:\, S_n^* < 0\}$ we obtain

$$\Psi(u) = \mathbf{P}(T_u < \infty) \ge \mathbf{P}(T_u^* < \infty)\,, \tag{5.2}$$

for any $u > 0$. Setting $Q_k^* = (\xi_k - \eta_k^*)/\lambda_k$ in (5.1), we represent S_n^* in the following form $S_n^* = \mathcal{E}_n\,(u - Y_n^*)$, where $Y_1^* = Q_1^*$ and for $n \ge 2$,

$$Y_n^* = Q_1^* + M_1 Q_2^* + \cdots + \prod_{j=1}^{n-1} M_j\, Q_n^*\,. \tag{5.3}$$

Therefore, for any $u > 0$,

$$\mathbf{P}(T_u^* < \infty) = \mathbf{P}(R^* > u)\,, \tag{5.4}$$

where $R^* = \sup_{n\ge1} Y_n^*$. To study the tail behaviour of R^* we need to obtain the renewal equation for R^*. To this end we rewrite Y_n^* as $Y_n^* = Q_1^* + M_1\, Z_n^*$ with $Z_2^* = Q_2^*$ and $Z_n^* = Q_2^* + M_2 Q_3^* + \cdots + \prod_{j=2}^{n-1} M_j\, Q_n^*$ for $n \ge 2$. By denoting $R_1 := \sup_{n\ge2} Z_n^*$ we get that $R^* = Q_1^* + M_1\,(R_1^*)_+$. Note that the random vector $(Z_2^*, \ldots, Z_n^*)$ has the same distribution as $(Y_1^*, \ldots, Y_{n-1}^*)$ for any $n \ge 2$, i.e. R^* has the same distribution as R_1 also. Moreover, taking into account that R_1^* is independent of (Q_1^*, M_1), we deduce that R^* satisfies the random Eq. (3.2). We show now that

$$\lim_{u\to\infty} u^{\beta}\, \mathbf{P}(R^* > u) = C_* > 0\,. \tag{5.5}$$

To prove this we check the conditions of Lemma 3.3. First, notice that (4.5) implies (3.5) for any $0 < \delta < \sqrt{\alpha_1 + \beta^2/4} - \beta/2$ with $\alpha_1 = 2\alpha/\sigma^2$. It easy to see that for such δ and any $x \in \mathbb{R}$ in this case $\mathbf{E}\, M_1^{\beta+\delta+ix} \ne 1$. Now we verify (3.7). Writing $q = \beta + \epsilon$ with $\epsilon > 0$, we obtain

$$\mathbf{E}\,|Q_1^*|^q \le \text{const}\,(\mathbf{E}\, M_1^q\, \mathbf{E}\, \xi_1^q + \mathbf{E}\,(\eta_1^*\, M_1)^q)\,.$$

By the conditions of Theorem 2.1 and (4.5) the first term in this inequality is finite for sufficiently small ϵ. Moreover, we prove that there exists $\varepsilon > 0$ such that

$\mathbf{E}\,(\eta_1^*\,M_1)^q < \infty$. Indeed, setting $w_u^* = \sup_{0\le s\le u}(-w_s - \frac{\kappa}{\sigma}\,s)$ we get

$$\begin{aligned}\mathbf{E}\,(\eta_1^*\,M_1)^q &= (c^*)^q\,\mathbf{E}\,\left(\int_0^{\theta_1} e^{-\sigma w_u - \kappa\,u}\,\mathrm{d}u\right)^q\\ &\le\ (c^*)^q\,\mathbf{E}\,\theta_1^q\,e^{q\sigma w^*_{\theta_1}} = (c^*)^q\,\alpha\,\int_0^\infty t^q\,\mathbf{E}\,e^{q\sigma w_t^*}\,e^{-\alpha t}\,\mathrm{d}t\,.\end{aligned}$$

The last intergal we estimate as

$$\int_0^\infty t^q\,\mathbf{E}\,e^{q\sigma w_t^*}\,e^{-\alpha t}\,\mathrm{d}t\ \le\ \frac{2}{\alpha}\,K_1^*\,\mathbf{E}\,e^{q\sigma w_\tau^*}\,,$$

where $K_1^* = \sup_{t\ge 0}(t^q\,e^{-\frac{\alpha}{2}t})$ and τ is an exponential random variable with the parameter $\alpha/2$ independing on $(w_u)_{u\ge 0}$. Moreover, taking into account that the random variable w_τ^* is exponential (see, for example, [2] p. 197) we find that

$$K_2^* = \mathbf{E}\,e^{q\sigma w_\tau^*} = \frac{\sqrt{\alpha\sigma^2+\kappa^2}+\kappa}{\sqrt{\alpha\sigma^2+\kappa^2}-\kappa-\varepsilon\sigma^2} < \infty$$

for $0 < \varepsilon\sigma^2 < \sqrt{\alpha\sigma^2+\kappa^2}-\kappa$. Therefore we get

$$\mathbf{E}\,(\eta_1^*\,M_1)^q \le 2(c^*)^q\,K_1^*\,K_2^*\,. \tag{5.6}$$

Now (5.5) follows from Lemma 3.3. Hence Theorem 2.2. $\square$

6 Exact asymptotics for the ruin probability

In this subsection we prove Theorem 2.3. For $\gamma = 0$, the theorem follows from (5.5). Therefore we assume $-\infty < \gamma < 0$. In this case Eq. (4.2) has the following form

$$S_n = \mathcal{E}_n u + \mathcal{E}_n \sum_{k=1}^n \mathcal{E}_k^{-1}(c_{k-1}\tilde{\eta}_k - \xi_k)\,, \tag{6.1}$$

where $c_n = c_{\tau_n} = c^*\exp\{\gamma\tau_n\}$ and $\tilde{\eta}_n = \int_0^{\theta_n} e^{h_{\tau_n} - h_{u+\tau_{n-1}} + \gamma u}\mathrm{d}u$. We set $\tilde{Y}_n := \sum_{k=1}^n \mathcal{E}_k^{-1} c_{k-1}\tilde{\eta}_k = \sum_{k=1}^n \prod_{j=1}^{k-1}\,\tilde{M}_j\,\tilde{Q}_k$ with $\tilde{Q}_k = c^*\,M_k\,\tilde{\eta}_k$ and $\tilde{M}_k = e^{\gamma\theta_k}M_k$. Taking into account that $(\tilde{Y}_n)$ is an increasing sequence, we put

$$\tilde{R} = \tilde{Y}_\infty = \lim_{n\to\infty}\,\tilde{Y}_n = \sum_{k=1}^\infty \prod_{j=1}^{k-1}\,\tilde{M}_j\,\tilde{Q}_k \quad \text{a.s.}$$

Notice now that this random variable satisfies the following identity in law

$$\tilde{R} \stackrel{(d)}{=} \tilde{Q} + \tilde{M}\tilde{R}\,,$$

where $\tilde{Q} \stackrel{(d)}{=} \tilde{Q}_1$, $\tilde{M} \stackrel{(d)}{=} \tilde{M}_1$ and $\tilde{R}$ is independent of $(\tilde{Q},\tilde{M})$. Moreover, for $q = \tilde{\beta} = \beta - 2\gamma/\sigma^2$ we get $\mathbf{E}\,\tilde{M}^q = \alpha(\alpha + (\tilde{\beta} - q)\,q\,\sigma^2/2)^{-1} = 1$ and similarly to (5.6) we can show that $\mathbf{E}\,\tilde{Q}^{\tilde{\beta}} < \infty$. Therefore, Lemma 3.2 implies that $\lim_{u\to\infty} u^{\tilde{\beta}}\,\mathbf{P}(\tilde{R} > u) < \infty$. Thus, by (4.9)

$$\lim_{u\to\infty} \frac{\mathbf{P}(\tilde{R} > u)}{\mathbf{P}(R > u)} = 0\,. \tag{6.2}$$

Now we study the stopping time (4.1) in our case. First, by (6.1) we may write T_u as

$$T_u := \inf\{n \geq 1:\ S_n < 0\} = \inf\{n \geq 1:\ Y_n > u + \tilde{Y}_n\}\,, \tag{6.3}$$

where Y_n is defined in (4.4).

Recall that, $R = Y_\infty = \lim_{n\to\infty} Y_n$ a.s. and $\tilde{R} = \tilde{Y}_\infty = \lim_{n\to\infty} \tilde{Y}_n$ a.s.. Therefore from (6.3) it follows that $\mathbf{P}(R > u + \tilde{R}\,,\, T_u = \infty) = 0$. Taking this into account, it easy to deduce the following equality

$$\mathbf{P}(T_u < \infty) = \mathbf{P}(Y_{T_u} > u + \tilde{Y}_{T_u}) = \mathbf{P}(R > u + \tilde{Y}_{T_u})\,. \tag{6.4}$$

From here we obtain for any $\delta > 0$,

$$\begin{aligned}
\mathbf{P}(T_u < \infty) &\geq \mathbf{P}(R > u + \tilde{Y}_{T_u},\, \tilde{Y}_{T_u} \leq \delta u) \geq \mathbf{P}(R > (1+\delta)u, \tilde{Y}_{T_u} \leq \delta u) \\
&= \mathbf{P}(R > (1+\delta)u) - \mathbf{P}(R > (1+\delta)u\,, \tilde{Y}_{T_u} > \delta\, u) \\
&\geq \mathbf{P}(R > (1+\delta)u) - \mathbf{P}(\tilde{R} > \delta\, u)\,.
\end{aligned}$$

The limiting relationships (4.9) and (6.2) imply that

$$\liminf_{u\to+\infty} \mathbf{P}(T_u < \infty)/\mathbf{P}(R > u) \geq 1\,.$$

Moreover, by (6.4) we obtain $\mathbf{P}(T_u < \infty) \leq \mathbf{P}(R > u)$ for any $u > 0$. Thus

$$\lim_{u\to\infty} \mathbf{P}(T_u < \infty)/\mathbf{P}(R > u) = 1\,.$$

If $\gamma = -\infty$, i.e. $c_t = 0$, then $\tilde{Y}_n = 0$ for all $n \in \mathbb{N}$ and , hence, $\mathbf{P}(T_u < \infty) = \mathbf{P}(R > u)$. Therefore (4.9) implies this theorem in this case. $\square$

7 Erdodic properties for the random coefficient autoregressive process

To show Theorem 2.4 we need to use some ergodic properties of the special autoregressive process with random coefficients (5.1). In this section we study the ergodic properties for a general scalar autoregressive process with random coefficient

$$x_n \,=\, a_n\, x_{n-1} \,+\, b_n\,,\; n \ge 1\,, \tag{7.1}$$

where x_0 is some fixed constant and $(a_n\,, b_n)$ is i.i.d. sequence of random variables in $\mathbb{R}^2$.

Proposition 7.1. *Assume that there exists $0 < \delta \le 1$ such that $\rho = \mathbf{E}\,|a_1|^\delta < 1$ and $\mathbf{E}\,|b_1|^\delta \,<\, \infty$. Then for any bounded uniformly continuous function f*

$$\mathbf{P} \,-\, \lim_{N\to\infty} N^{-1} \sum_{n=1}^{N} f(x_n) \,=\, \mathbf{E}\, f(x_\infty)\,, \tag{7.2}$$

where $x_\infty = \sum_{k=1}^{\infty} \pi_{k-1} b_k$ with $\pi_0 = 1$ and $\pi_k = \prod_{j=1}^{k} a_j$ for $k \ge 1$.

Proof. First we show that the series in the definition of x_∞ converges in probability. Indeed, $\mathbf{E}\,|\sum_{k=n}^{n+m} \pi_{k-1}\, b_k|^\delta \;\le\; \mathbf{E}\,|b_1|^\delta \sum_{k=n}^{n+m} \rho^k$. It means that the series $\sum_{k\ge1} \pi_{k-1}\, b_k$ converges in $\mathbf{L}_\delta$ and hence in probability. Now we fixe some $m \ge 1$ and, for $n \ge m$, we set $x_n(m) \,=\, \sum_{k=n-m+1}^{n} b_k \prod_{j=k+1}^{n} a_j$. Notice that $x_n(m)$ is mesurable with respect to $\sigma\{a_{n-m+1}, \ldots, a_n, b_{n-m+1}, \ldots, b_n\}$. Therefore for any $0 \le d < m$ the the sequence $(x_{km+d})_{k\ge1}$ is i.i.d. and by the law of large numbers for any fixed $m \ge 1$ and $0 \le d < m$

$$\lim_{p\to\infty} p^{-1} \sum_{k=1}^{p} f(x_{km+d}(m)) \,=\, \mathbf{E}\, f(x_m(m)) \quad \text{a.s.}\,, \tag{7.3}$$

where $x_m(m) = \sum_{k=1}^{m} b_k \prod_{j=k+1}^{m} a_j \overset{(d)}{=} \sum_{k=1}^{m} b_k\, \pi_{k-1}$. Therefore

$$\lim_{m\to\infty} \mathbf{E}\, f(x_m(m)) \,=\, \mathbf{E}\, f(x_\infty)\,. \tag{7.4}$$

We show now that for any $\epsilon > 0$

$$\lim_{m\to\infty} \sup_{N\ge m} \mathbf{P}(\, \Delta(N,m) \,>\, \epsilon\,) \,=\, 0\,, \tag{7.5}$$

where $\Delta(N,m) = N^{-1} \sum_{n=m}^{N} |f(x_n) - f(x_n(m))|$.

We put $x_n^*(m) = x_n - x_n(m) = x_{n-m} \prod_{k=n-m+1}^n a_j$. Taking into account that there exists some $L^* < \infty$ such that for any $n \ge 1$

$$\mathbf{E}\,|x_n|^\delta = \mathbf{E}\,|x_0 \prod_{k=1}^n a_j + \sum_{k=2}^n b_k \prod_{j=k+1}^n a_j\,|^\delta$$

$$\le |x_0|^\delta \rho^n + \mathbf{E}\,|b_1|^\delta \sum_{k=2}^n \rho^{n-k} \le L^*\,,$$

we get $\sup_{n\ge m} \mathbf{E}\,|x_n^*(m)|^\delta \le L^* \rho^m$.

Let us choose $\epsilon_1 > 0$ for which $\sup_{|x-y|\le\epsilon_1} |f(x) - f(y)| \le \epsilon/2$. For such ϵ_1 we obtain that $\Delta(N,m) \le \epsilon/2 + 2f^* N^{-1} \sum_{n=m}^N \mathbf{1}_{\{|x_n^*(m)|\ge\epsilon_1\}}$, where $f^* = \sup_{x\in\mathbb{R}} |f(x)|$. Therefore by denoting $\epsilon^* = \epsilon/4f^*$ we get that

$$\mathbf{P}(\,\Delta(N,m) > \epsilon\,) \le \mathbf{P}\left(\sum_{n=m}^N \mathbf{1}_{\{|x_n^*(m)|\ge\epsilon_1\}} > \epsilon^* N\right)\,.$$

Applying here the Chebyshev inequality we find that

$$\mathbf{P}(\,\Delta(N,m) > \epsilon\,) \le \frac{1}{\epsilon^* N} \sum_{n=m}^N \mathbf{P}(|x_n^*(m)| \ge \epsilon_1) \le L^* \frac{1}{\epsilon_1^\delta\, \epsilon^*} \rho^m\,.$$

This implies (7.5). We put $p = [N/m]$ ($[a]$ is the whole part of a), i.e. $N = pm + r$ with $0 \le r < m$). For such p and r, we can write that

$$\Omega_N := \left|\frac{1}{N}\sum_{n=1}^N f(x_n) - \mathbf{E}\,f(x_\infty)\right| \le \left|\frac{1}{N}\sum_{n=m}^{pm-1} f(x_n(m)) - \mathbf{E}\,f(x_\infty)\right|$$

$$+ f^* \frac{m+r}{N} + \Delta(N,m)\,.$$

Moreover, we can represent the last sum in this inequality as

$$\sum_{n=m}^{pm-1} f(x_n(m)) = \sum_{d=0}^{m-1} \sum_{k=1}^{p-1} f(x_{km+d}(m))\,.$$

Therefore, from (7.3), we get that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=m}^{pm-1} f(x_n(m)) = \lim_{p\to\infty} \frac{1}{m} \sum_{d=0}^{m-1} \frac{1}{p} \sum_{k=1}^{p-1} f(x_{km+d}(m)) = \mathbf{E}\,f(x_m(m))\,.$$

Finally, for $\epsilon > 0$, we obtain that for any $m \ge 1$

$$\limsup_{N\to\infty} \mathbf{P}\,(\,\Omega_N > \epsilon\,) \le \sup_{N\ge 1} \mathbf{P}\,(\,|\mathbf{E}\,f(x_\infty) - \mathbf{E}\,f(x_m(m))| + \Delta(N,m) > \epsilon\,)\,.$$

The limiting relationships (7.4)–(7.5) imply (7.2). $\square$

8 Large volatility

In this section we prove Theorem 2.4. First, notice that if $\beta < 0$ then Proposition 4 in [5] implies that $\mathbf{P}(T_u^* < \infty) = 1$ for any $u \geq 0$. Thus Theorem 2.4 for $\beta < 0$ directly follows from Inequality (5.2). We consider the critical case $\beta = 0$, i.e. $\kappa = 0$ and $\lambda_k = e^{\sigma \nu_k}$ with $\nu_k = w^k_{\theta_k} = w_{\tau_k} - w_{\tau_{k-1}}$.

For this, we study the ergodic properties of the process (S_n^*) defined in (5.1). Notice that (5.1) implies that this process satifies the following random reccurence equation

$$S_n^* = \lambda_n \, S_{n-1}^* \, + \, \zeta_n^* \,, \tag{8.1}$$

where $S_0^* = u$ and ζ_n^* is defined in (5.1).

Set $t_0 = 0$ and $t_n = \inf\{k > t_{n-1} \, : \, \sum_{j=t_{n-1}+1}^{k} \nu_j < 0\}$ for $n \geq 1$. It is easy to see that $t_n = \sum_{j=1}^{n} \rho_j$, where (ρ_j) is an i.i.d. sequence which has the same distribution as t_1 whose properties are well known, see XII. 7 theorem 1a in [4]. One can show, that for some constant $0 < c < \infty$,

$$\sup_{n\geq 1} n^{1/2} \mathbf{P}(t_1 > n) \leq c \,. \tag{8.2}$$

Set $x_n^* = S_{t_n}^*$. By (8.1) we obtain that for any $n \geq 1$,

$$x_n = a_n \, x_{n-1} + b_n \,, \quad x_0 = u \,, \tag{8.3}$$

where $a_n = \prod_{j=1}^{\rho_n} \lambda_{t_{n-1}+j} = \exp\{\sigma \sum_{j=1}^{\rho_n} \nu_{t_{n-1}+j}\}$ and

$$b_n = \sum_{k=1}^{\rho_n} \big(\prod_{j=k+1}^{\rho_n} \lambda_{t_{n-1}+j} \big) \, \zeta^*_{t_{n-1}+k} \,.$$

The sequence (a_n, b_n) is an i.i.d. sequence of random variables in $\mathbb{R}^2$. Moreover, $\mathbf{E}\, a_n = \mathbf{E}\, a_1 < 1$. We will show that there exists $r > 0$ such that

$$\mathbf{E}\, |b_1|^r < \infty \,. \tag{8.4}$$

First, notice that the definition of b_1 implies that $|b_1| \leq \sum_{k=1}^{t_1} |\zeta_k^*|$. Moreover, similarly to (5.6) we can show that there exists $0 < \epsilon < 1$ for which $\mathbf{E}\, |\eta_1^*|^\epsilon < \infty$. Therefore, taking into account the condition of Theorem 2.4 ($\mathbf{E}\, \xi_1^\delta < \infty$ for some $\delta > 0$) we get that there exists $0 < \epsilon < 1$ such that $m_\epsilon = \mathbf{E}\, |\zeta_1^*|^\epsilon < \infty$. To finish the proof of inequality (8.4), note that, for such ϵ and for some fixed $0 < r < 1$, by

making use of inequality (8.2) we obtain that

$$
\begin{aligned}
\mathbf{E}\,|b_1|^r &\le 1 + r\sum_{n=1}^{\infty}\frac{1}{n^{1-r}}\,\mathbf{P}(\sum_{k=1}^{t_1}|\zeta_k^*|>n)\\
&\le 1 + r\sum_{n=1}^{\infty}\frac{1}{n^{1-r}}\,\mathbf{P}(\sum_{k=1}^{l_n}|\zeta_k^*|>n) + r\sum_{n=1}^{\infty}\frac{1}{n^{1-r}}\,\mathbf{P}(t_1>l_n)\\
&\le 1 + r\,m_\epsilon\sum_{n=1}^{\infty}\frac{l_n}{n^{1-r+\epsilon}} + r\,c\sum_{n=1}^{\infty}\frac{1}{n^{1-r}l_n^{1/2}}\,.
\end{aligned}
$$

Therefore, by putting $l_n = [n^{4r}]$, we obtain (8.4) for $0 < r < \epsilon/5$. Hence, by Proposition 7.1, the process (8.3) has the property (7.2) for some bounded uniform continuous function f.

For Eq. (8.3) we reprsent the random variable $x_\infty = \sum k \ge 1\pi_{k-1}b_k$ as $x_\infty := \prod_{j=2}^{t_1}\lambda_j(\varsigma-\xi_1)$, where ς is independent of ξ_1. This implies that $\mathbf{P}(x_\infty^* < 0) = \mathbf{P}(\xi_1 > \varsigma)$. Thus, by the condition on the distribution of ξ_1 we obtain that $\mathbf{P}(x_\infty^* < 0) > 0$. It means that for the function $f_1(x) = \min(x^2,1)\mathbf{1}_{\{x\le 0\}}$ we have $\mathbf{E}\,f_1(x_\infty) > 0$ and by (7.2) there exists a sequence (n_k) such that $\lim_{k\to\infty} n_k^{-1}\sum_{j=1}^{n_k} f_1(x_j) = \mathbf{E}\,f_1(x_\infty) > 0$ a.s. Therefore $\mathbf{P}(T_u^* < \infty) = 1$ and Theorem 2.3 follows directly from (5.2). $\square$

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